

Optimality conditions for the buckling of a clamped plate.

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Abstract

We prove the following uniqueness result for the buckling plate. Assume there exists a smooth domain which minimizes the first buckling eigenvalue for a plate among all smooth domains of given volume. Then the domain must be a ball. The proof uses the second variation for the buckling eigenvalue and an inequality by L. E. Payne to establish this result.

Key words: buckling load, fourth order, second shape derivative.

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1 Introduction

We consider the following variational problem. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and let

$$\mathcal{R}(u, \Omega) := \frac{\int_{\Omega} |\Delta u|^2 dx}{\int_{\Omega} |\nabla u|^2 dx}$$

for $u \in H_0^{2,2}(\Omega)$. We set $\mathcal{R}(u, \Omega) = \infty$ if the denominator vanishes. We define

$$(1.1) \quad \Lambda(\Omega) := \inf \left\{ \mathcal{R}(u, \Omega) : u \in H_0^{2,2}(\Omega) \right\}.$$

The infimum is attained by the first eigenfunction u , which solves the Euler Lagrange equation

$$(1.2) \quad \Delta^2 u + \Lambda(\Omega) \Delta u = 0 \quad \text{in } \Omega$$

$$(1.3) \quad u = \partial_\nu u = 0 \quad \text{in } \partial\Omega.$$

If we normalize u by $\|\nabla u\|_{L^2(\Omega)} = 1$, the first eigenfunction is uniquely determined. Otherwise any multiple of u is an eigenfunction as well. The sign of the first eigenfunction may change depending on Ω .

The quantity $\Lambda(\Omega)$ is called buckling eigenvalue of Ω . It is well known that there is a discrete spectrum of positive eigenvalues of finite multiplicity and their only accumulation point is ∞ . The corresponding eigenfunctions form an orthonormal basis of $H_0^{2,2}(\Omega)$.

In the sequel, we will assume that u is normalized. If we multiply (1.2) with $x \cdot \nabla u$ and integrate by parts, we obtain

$$(1.4) \quad \Lambda(\Omega) = \frac{1}{2} \int_{\partial\Omega} |\Delta u|^2 x \cdot \nu \, dS.$$

In 1951, G. Polya and G. Szegő formulated the following conjecture (see [9]).

Among all domains Ω of given volume, the ball minimizes $\Lambda(\Omega)$.

This conjecture is still open. However, partial results are known. In [11] Szegő proved the conjecture for all smooth domains under the additional assumption that $u > 0$ in Ω . M.S. Ashbaugh and D. Bucur proved that among simply connected domains of prescribed volume there exists an optimal domain [1]. In [12] H. Weinberg and B. Willms proved the following uniqueness result for $n = 2$. If an optimal plane domain Ω exists and if $\partial\Omega$ is smooth (at least $C^{2,\alpha}$), then Ω is a disc.

There also exist bounds for $\Lambda(\Omega)$. We only mention Payne's inequality (see [13]) which states that

$$\Lambda(\Omega) \geq \lambda_2(\Omega),$$

where λ_2 denotes the second Dirichlet eigenvalue for the Laplacian. Equality holds if and only if Ω is a ball.

In this paper, we assume that there exists an optimal domain Ω , which is smooth and simply connected. We will prove that Ω must be a ball. Thus we generalize the result of H. Weinberg and B. Willms in [12] to higher dimensions.

To consider the second domain variation for $\Lambda(\Omega)$ is motivated by the work of E. Mohr in [6]. He was interested in the clamped plate eigenvalue, where

$$\mathcal{R}(u, \Omega) = \frac{\int_{\Omega} |\Delta u|^2 dx}{\int_{\Omega} u^2 dx}$$

and Ω is a smoothly bounded domain in \mathbb{R}^2 . For the corresponding eigenvalue he computed the second domain variation. The explicit computation of the kernel of the second domain variation then implies that the disc is a unique minimizer among smooth domains of equal volume.

Our strategy will be as follows. In Chapter 2 we introduce a smooth family $(\Omega_t)_t$ of perturbations of Ω of equal volume. We denote by $\Lambda(t) := \Lambda(\Omega_t)$ the corresponding first buckling eigenvalue of Ω_t . As a consequence of the optimality of Ω , the eigenfunction u satisfies the overdetermined boundary value problem

$$\begin{aligned} \Delta^2 u + \Lambda(\Omega) \Delta u &= 0 \text{ in } \Omega \\ u = \partial_{\nu} u &= 0 \text{ in } \partial\Omega \\ \Delta u &= c_0 \text{ in } \partial\Omega, \text{ where } c_0 = \frac{2\Lambda(\Omega)}{|\Omega|} \text{ by (1.4)}. \end{aligned}$$

This follows from the fact that the first domain variation of $\Lambda(\Omega)$ - computed in Chapter 3 - for an optimal domain necessarily vanishes.

In Chapter 4 we compute the second domain variation of $\Lambda(\Omega)$. It turns out that

$$(1.5) \quad \ddot{\Lambda}(0) = \frac{d^2}{dt^2} \Lambda(t) \Big|_{t=0} = 2 \int_{\Omega} |\Delta u'|^2 - 2\Lambda(\Omega) \int_{\Omega} |\nabla u'|^2 dx,$$

where u' is the so called shape derivative of u . It solves

$$(1.6) \quad \Delta^2 u' + \Lambda(\Omega) \Delta u' = 0 \text{ in } \Omega$$

$$(1.7) \quad u' = 0 \text{ in } \partial\Omega$$

$$(1.8) \quad \partial_{\nu} u' = -c_0 v \cdot \nu \text{ in } \partial\Omega$$

and

$$(1.9) \quad \int_{\Omega} \nabla u \cdot \nabla u' dx = 0.$$

The vector field v is the first order approximation of Ω_t in the sense that for $y \in \Omega_t$ there exists an $x \in \Omega$ such that

$$y = x + tv(x) + o(t).$$

Thus, $\ddot{\Lambda}(0)$ is equal to a quadratic functional in the shape derivative u' which we denote by $\mathcal{E}(u')$ and $\mathcal{E}(u')$ is given by the right hand side of (1.5). Since we assume the optimality of Ω , we have $\mathcal{E}(u') \geq 0$. It turns out that the kernel of $\mathcal{E}(u')$ contains the directional derivatives $\partial_1 u, \dots, \partial_n u$ of u . Each directional derivative is a shape derivative, which corresponds to a domain perturbation given by translations.

The key idea is to enlarge the class of shape derivatives on which \mathcal{E} is defined. This new class will be denoted by \mathcal{Z} and contains the shape derivatives as a true subset. Nevertheless we can show that \mathcal{E} is still bounded from below and even nonnegative on \mathcal{Z} . Moreover $\min_{\mathcal{Z}} \mathcal{E} = 0$ since the directional derivatives of u are in \mathcal{Z} . This is done in Chapter 5. In Chapter 6 we construct a function $\psi \in \mathcal{Z}$ for which we will show

$$0 \leq \mathcal{E}(\psi) \leq (\lambda_2(\Omega) - \Lambda(\Omega)) \lambda_2(\Omega).$$

By Payne's inequality we have equality and this proves that the optimal domain is a ball.

Some of these results were obtained in the Diplom thesis of the first author [5].

2 Domain variations

Let Ω be a bounded smooth (at least $C^{2,\alpha}$) and simply connected domain in \mathbb{R}^n . We denote by ν the unit normal vector field on $\partial\Omega$. Let δ be the distance function to the boundary, i.e. for $x \in \bar{\Omega}$ we have

$$\delta(x) := \inf\{|x - z| : z \in \partial\Omega\}.$$

Then, for smooth $\partial\Omega$, $\nu := \nabla\delta$ defines a smooth extension of ν into a sufficiently small tubular neighbourhood of $\partial\Omega$. With this the following identities hold.

$$(2.1) \quad \nu \cdot \nu = 1, \quad \nu \cdot D\nu = 0 \quad \text{and} \quad D\nu \cdot \nu = 0$$

on $\partial\Omega$. See e.g. Proposition 5.4.14 in [4] for a proof.

Moreover, the mean curvature of $\partial\Omega$ is bounded since Ω is smooth, i.e. for each $x \in \partial\Omega$ there holds

$$(2.2) \quad |H_{\partial\Omega}(x)| \leq \max_{\partial\Omega} |H_{\partial\Omega}| < \infty.$$

We will frequently use integration by parts on $\partial\Omega$. Let $f \in C^1(\partial\Omega)$ and $v \in C^{0,1}(\partial\Omega, \mathbb{R}^n)$. The next formula is often called the *Gauss theorem on surfaces*.

$$(2.3) \quad \oint_{\partial\Omega} f \operatorname{div}_{\partial\Omega} v \, dS = - \oint_{\partial\Omega} v \cdot \nabla^\tau f \, dS + (n-1) \oint_{\partial\Omega} f(v \cdot \nu) H_{\partial\Omega} \, dS,$$

where

$$(2.4) \quad \nabla^\tau f = \nabla f - (\nabla f \cdot \nu)\nu$$

denotes the tangential gradient of f .

In this chapter, we describe the class of admissible variations for the domain functional $\Lambda(\Omega)$. For given $t_0 > 0$ and $t \in (-t_0, t_0)$ let $(\Omega_t)_t$ be a family of perturbations of the domain $\Omega \subset \mathbb{R}^n$ of the form

$$\Omega_t = \Phi_t(\Omega)$$

where

$$\Phi_t : \overline{\Omega} \rightarrow \mathbb{R}^n$$

is a diffeomorphism which is smooth in t and x . Thus we may write

$$\Omega_t := \{y = x + tv(x) + \frac{t^2}{2}w(x) + o(t^2) : x \in \Omega, t \text{ small}\},$$

where

$$v = (v_1(x), v_2(x), \dots, v_n(x)) = \partial_t \Phi_t(x)|_{t=0}$$

and

$$w = (w_1(x), w_2(x), \dots, w_n(x)) = \partial_t^2 \Phi_t(x)|_{t=0}$$

are smooth vector fields and where $o(t^2)$ collects terms such that $\frac{o(t^2)}{t^2} \rightarrow 0$ as $t \rightarrow 0$. For small t_0 the sets Ω_t and Ω are diffeomorphic. We will frequently use the notation $y := \Phi_t(x)$. Consider the functional

$$\Lambda(\Omega_t) := \inf \left\{ \mathcal{R}(u, \Omega_t) : u \in H_0^{2,2}(\Omega_t) \right\},$$

which only depends on Ω_t . Let $u(t, y) \in H_0^{2,2}(\Omega_t)$ be the minimizer. For short we will write

$$(2.5) \quad \tilde{u}(t) := u(t, y).$$

Then $\tilde{u}(t)$ solves

$$(2.6) \quad \Delta^2 \tilde{u}(t) + \Lambda(\Omega_t) \Delta \tilde{u}(t) = 0 \quad \text{in } \Omega_t$$

$$(2.7) \quad \tilde{u}(t) = |\nabla \tilde{u}(t)| = 0 \quad \text{in } \partial\Omega_t.$$

for each $t \in (-t_0, t_0)$. With this notation we define

$$\Lambda(t) := \mathcal{R}(\tilde{u}(t), \Omega_t).$$

Since we assume smoothness of Ω and Φ_t the eigenfunction \tilde{u} is also smooth in t and x . This has several consequences which we list as remarks.

Remark 1 Since $\partial\Omega_t$ is smooth and since $\tilde{u}(t) = 0$ on $\partial\Omega_t$ then necessarily

$$(2.8) \quad \Delta\tilde{u} = \partial_\nu^2\tilde{u} + (n-1)\partial_\nu\tilde{u} H_{\partial\Omega_t} \quad \text{in } \partial\Omega_t,$$

where $H_{\partial\Omega_t}$ denotes the mean curvature of $\partial\Omega_t$. Clearly, if $\tilde{u} = |\nabla\tilde{u}| = 0$ on $\partial\Omega_t$, then necessarily

$$(2.9) \quad \Delta\tilde{u} = \partial_\nu^2\tilde{u} \quad \text{in } \partial\Omega_t.$$

Remark 2 Since (2.7) holds for all $t \in (-t_0, t_0)$, we also have

$$(2.10) \quad \dot{\tilde{u}}(t) = |\nabla\dot{\tilde{u}}(t)| = 0 \quad \text{in } \partial\Omega_t$$

for all $t \in (-t_0, t_0)$.

Remark 3 Straightforward computations yield

$$\dot{\tilde{u}}(t) = \frac{d}{dt}u(t, y) = \partial_t u(t, \Phi_t(\Phi_t^{-1}(y))) + \partial_t \Phi_t(\Phi_t^{-1}(y)) \cdot \nabla u(t, y)$$

for all $t \in (-t_0, t_0)$. Let $y \in \partial\Omega_t$. Then (2.10) and (2.7) imply

$$(2.11) \quad 0 = \dot{\tilde{u}}(t) = \partial_t u(t, y) \quad \text{for } y \text{ in } \partial\Omega_t$$

for all $t \in (-t_0, t_0)$.

In particular for $t = 0$ we compute $\tilde{u}(0) = u(x)$ and

$$\begin{aligned} \dot{\tilde{u}}(0) &= \partial_t u(0, x) + v(x) \cdot Du(0, x) \\ \ddot{\tilde{u}}(0) &= \partial_t^2 u(0, x) + 2v(x) \cdot D\partial_t u(0, x) + w(x) \cdot Du(0, x) + v(x) \cdot D(v(x) \cdot Du(0, x)). \end{aligned}$$

We will use the notation

$$u'(x) := \partial_t u(0, x) \quad \text{and} \quad u''(x) := \partial_t^2 u(0, x).$$

Hence,

$$(2.12) \quad \dot{\tilde{u}}(0) = u'(x) + v(x) \cdot Du(x)$$

$$(2.13) \quad \ddot{\tilde{u}}(0) = u''(x) + 2v(x) \cdot Du'(x) + w(x) \cdot Du(x) + v(x) \cdot D(v(x) \cdot Du(x)).$$

Note that all these quantities are defined for $x \in \overline{\Omega}$. For $x \in \partial\Omega$ we thus get

$$0 = \dot{\tilde{u}}(0) = u'(x) \quad \text{and} \quad 0 = \nabla\dot{\tilde{u}}(0) = \nabla u'(x) + v(x) \cdot D^2 u(x),$$

where $(v(x) \cdot D^2 u(x))_j = \sum_{i=1}^n v_i(x) \partial_i \partial_j u(x)$ for $j = 1, \dots, n$. Thus, we get the following boundary conditions for u' .

$$(2.14) \quad u'(x) = 0 \quad \text{and} \quad \partial_\nu u'(x) = -v(x) \cdot D^2 u(x) \cdot \nu(x) \quad \text{for } x \in \partial\Omega.$$

Here we used the notation $v(x) \cdot D^2u(x) \cdot \nu(x) = \sum_{i,j=1}^n v_i(x) \partial_i \partial_j u(x) \nu_j(x)$.

Let $\nu_t(y)$ be the unit normal vector in $y \in \partial\Omega_t$. We also write this as

$$(2.15) \quad \nu_t(y) = \nu(t, \Phi_t(x)) \quad \forall t \in (-t_0, t_0) \quad x \in \partial\Omega.$$

Then we have

$$(2.16) \quad \nu' = -\nabla^\tau(v \cdot \nu), \quad \nu \cdot \nu' = 0.$$

This follows from direct calculations (see e.g. (5.64) in [4]).

Lemma 1 *With the notation from above the following equality holds.*

$$(2.17) \quad \nu_t \cdot \nabla(\partial_t u(t, y)) = -\Delta u(t, y) \nu_t \cdot \partial_t \Phi_t(\Phi_t^{-1}(y)) \quad \text{for } y \text{ in } \partial\Omega_t$$

for all $t \in (-t_0, t_0)$. Alternatively, we write this for all $t \in (-t_0, t_0)$ and $x \in \partial\Omega$ as

$$(2.18) \quad \nu(t, \Phi_t(x)) \cdot \nabla\{\partial_t u(t, \Phi_t(x))\} = -\Delta u(t, \Phi_t(x)) \nu(t, \Phi_t(x)) \cdot \partial_t \Phi_t(x).$$

Proof Since $\nabla u(t, \Phi_t(x)) = 0$ for all $|t| < t_0$ and all $x \in \partial\Omega$ we have

$$0 = \frac{d}{dt} \nabla u(t, \Phi_t(x)) = \nabla \partial_t u(t, \Phi_t(x)) + D^2 u(t, \Phi_t(x)) \cdot \partial_t \Phi_t(x).$$

This implies

$$0 = \nu_t \cdot \nabla(\partial_t u(t, y)) + \nu_t \cdot D^2 u(t, y) \cdot \partial_t \Phi_t(\Phi_t^{-1}(y)) \quad \text{for } y \text{ in } \partial\Omega_t$$

for all $t \in (-t_0, t_0)$. Here we used the notation

$$\nu_t \cdot D^2 u(t, y) \cdot \partial_t \Phi_t(\Phi_t^{-1}(y)) = \sum_{ij=1}^n \nu_{t,i} \cdot \partial_i \partial_j u(t, y) \cdot \partial_t \Phi_t(\Phi_t^{-1}(y))_j.$$

Since $\nabla \tilde{u}(t) = 0$ in $\partial\Omega_t$, we get

$$\nu_t \cdot D^2 u(t, y) \cdot \partial_t \Phi_t(\Phi_t^{-1}(y)) = \nu_t \cdot D^2 u(t, y) \cdot \nu_t \nu_t \cdot \partial_t \Phi_t(\Phi_t^{-1}(y)).$$

Thus,

$$\nu_t \cdot \nabla(\partial_t u(t, y)) = -\nu_t \cdot D^2 u(t, y) \cdot \nu_t \nu_t \cdot \partial_t \Phi_t(\Phi_t^{-1}(y)) \quad \text{for } y \text{ in } \partial\Omega_t.$$

Formula (2.9) simplifies to

$$\nu_t \cdot \nabla(\partial_t u(t, y)) = -\Delta u(t, y) \nu_t \cdot \partial_t \Phi_t(\Phi_t^{-1}(y)) \quad \text{for } y \text{ in } \partial\Omega_t.$$

This proves the lemma. □

The first derivative of $\Lambda(t)$ with respect to the parameter t is called *the first domain variation* and the second derivative is called *the second domain variation*.

Our domain variations will be chosen within the class of volume preserving perturbations up to order 2. Hence, they are chosen such that

$$(2.19) \quad \mathcal{L}^n(\Omega_t) = \mathcal{L}^n(\Omega) + o(t^2)$$

holds. This puts constraints on the vector fields v and w . They were discussed e.g. in [2], formula (2.13) and Lemma 1.

Lemma 2 *Let $v, w \in C^{0,1}(\Omega, \mathbb{R}^n)$ be such that (2.19) holds. Then*

$$(2.20) \quad \int_{\Omega} \operatorname{div} v \, dx = 0$$

and

$$\int_{\Omega} ((\operatorname{div} v)^2 - Dv : Dv + \operatorname{div} w) \, dx = 0,$$

where $Dv : Dv = \sum_{i,j=1}^n \partial_i v_j \, \partial_j v_i$. The second equality is equivalent to

$$(2.21) \quad \int_{\partial\Omega} (v \cdot \nu) \operatorname{div} v \, dS - \int_{\partial\Omega} v \cdot Dv \cdot \nu \, dS + \int_{\partial\Omega} (w \cdot \nu) \, dS = 0.$$

Note that rotations do not satisfy these conditions (see e.g. Remark 1 in [2]).

3 The first domain variation

We will use the following formula for the computations of the first domain variation of Λ . It is well known as Reynolds transport theorem and is analyzed in detail in Chapter 5.2.3 in [4].

Theorem 1 *Let $t \in (-t_0, t_0)$ for some $t_0 > 0$. Let $\Phi_t \in C^{0,1}(\mathbb{R}^n)$ be differentiable in t and let $t \rightarrow f(t) \in L^1(\mathbb{R}^n)$ be a function which is differentiable in t . Moreover, let $f(t) \in W^{1,1}(\mathbb{R}^n)$. Then $t \rightarrow I(t) := \int_{\Omega_t} f(t) \, dy$ is differentiable in t . Moreover, we have the formula*

$$\dot{I}(t) = \int_{\Omega_t} \partial_t f(t) + \operatorname{div} (f(t) \partial_t \Phi_t(\Phi_t^{-1}(y))) \, dy.$$

If $\partial\Omega$ is sufficiently smooth (at least Lipschitz continuous), this is equivalent to

$$\dot{I}(t) = \int_{\Omega_t} \partial_t f(t) \, dy + \int_{\partial\Omega_t} f(t) \partial_t \Phi_t(\Phi_t^{-1}(y)) \cdot \nu(y) \, dS(y).$$

In particular, for $t = 0$ we get

$$\dot{I}(0) = \int_{\Omega} \partial_t f(t)|_{t=0} + \operatorname{div} (f(0) v(x)) \, dx.$$

Again, if $\partial\Omega$ is sufficiently smooth, this is equivalent to

$$\dot{I}(0) = \int_{\Omega} \partial_t f(t)|_{t=0} \, dx + \int_{\partial\Omega} f(0) v(x) \cdot \nu(x) \, dS(x).$$

We apply this formula to $\Lambda(t) = \frac{D(t)}{N(t)}$ where

$$D(t) := \int_{\Omega_t} |\Delta \tilde{u}(t)|^2 \, dy \quad \text{and} \quad N(t) := \int_{\Omega_t} |\nabla \tilde{u}(t)|^2 \, dy$$

and we assume the normalization

$$(3.1) \quad N(t) = \int_{\Omega_t} |\nabla \tilde{u}(t)|^2 \, dy = 1 \quad \forall t \in (-t_0, t_0).$$

We then obtain

$$\begin{aligned} \dot{\Lambda}(t) &= 2 \int_{\Omega_t} \Delta \tilde{u}(t) \Delta \partial_t \tilde{u}(t) \, dy - 2 \Lambda(t) \int_{\Omega_t} \nabla \tilde{u}(t) \cdot \nabla \partial_t \tilde{u}(t) \, dy \\ &\quad + \int_{\partial\Omega_t} |\Delta \tilde{u}(t)|^2 \partial_t \Phi_t(\Phi_t^{-1}(y)) \cdot \nu_t(y) \, dS(y), \end{aligned}$$

where $\nu_t(y)$ denotes the unit normal vector in $y \in \partial\Omega_t$. We integrate by parts and use (2.10). Then

$$\begin{aligned} \dot{\Lambda}(t) &= 2 \int_{\Omega_t} \{ \Delta^2 \tilde{u}(t) + \Lambda(t) \Delta \tilde{u}(t) \} \partial_t \tilde{u}(t) \, dy + 2 \int_{\partial\Omega_t} \Delta \tilde{u}(t) \partial_{\nu_t} \partial_t \tilde{u}(t) \, dS(y) \\ &\quad - 2 \int_{\partial\Omega_t} \partial_{\nu_t} \Delta \tilde{u}(t) \partial_t \tilde{u}(t) \, dS(y) + \int_{\partial\Omega_t} |\Delta \tilde{u}(t)|^2 \partial_t \Phi_t(\Phi_t^{-1}(y)) \cdot \nu_t(y) \, dS(y). \end{aligned}$$

The first integral vanishes since $\tilde{u}(t)$ solves (2.6). The third integral vanishes since (2.11) holds. Finally we use (2.17). This proves the following lemma.

Lemma 3 *Let $\tilde{u}(t)$ be an eigenfunction (i.e. a solution of (2.6) - (2.7)) and assume (3.1) holds. Let*

$$\Lambda(t) = \int_{\Omega_t} |\Delta \tilde{u}(t)|^2 \, dy.$$

Then

$$(3.2) \quad \dot{\Lambda}(t) = - \int_{\partial\Omega_t} |\Delta \tilde{u}(t)|^2 \partial_t \Phi_t(\Phi_t^{-1}(y)) \cdot \nu_t(y) dS(y).$$

Remark 4 Note that if $\partial_t \Phi_t(\Phi_t^{-1}(y)) \cdot \nu_t(y) > 0$, this implies $\mathcal{L}^n(\Omega_t) > \mathcal{L}^n(\Omega)$ for small t . Thus, $\dot{\Lambda}(t)$ is negative in this case. From this we conclude that the first buckling eigenvalue is decreasing under set inclusion.

From Lemma 3 we get in particular

$$\dot{\Lambda}(0) = - \int_{\partial\Omega} |\Delta u|^2 v(x) \cdot \nu(x) dS(x).$$

From Lemma 2 and (2.20) we deduce $|\Delta u| = \text{const.}$ if Ω is a critical point of $\Lambda(t)$. Due to formula (1.4), this constant is equal to

$$(3.3) \quad c_0 := \frac{2\Lambda(0)}{|\Omega|}.$$

We denote this result as a theorem.

Theorem 2 Let Ω_t be a family of volume preserving perturbations of Ω as described in Chapter 2. Then Ω is a critical point of the energy $\Lambda(t)$, i.e. $\dot{\Lambda}(0) = 0$, if and only if

$$(3.4) \quad \Delta u = c_0 \quad \text{on } \partial\Omega.$$

In particular, u is a solution of the overdetermined boundary value problem

$$(3.5) \quad \Delta^2 u + \Lambda(\Omega) \Delta u = 0 \quad \text{in } \Omega$$

$$(3.6) \quad u = \partial_\nu \nabla u = 0 \quad \text{in } \partial\Omega.$$

$$(3.7) \quad \Delta u = c_0 > 0 \quad \text{in } \partial\Omega.$$

Note that if we set $U := \Delta u + \Lambda(\Omega)u$ (3.5) - (3.7) implies

$$\Delta U = 0 \text{ in } \Omega \quad \text{and} \quad U = c_0 \text{ in } \partial\Omega.$$

Hence,

$$(3.8) \quad U = \Delta u + \Lambda(\Omega)u = c_0 \quad \text{in } \overline{\Omega}.$$

From [12] we know that for $n = 2$ this implies that Ω is a ball. In particular,

$$(3.9) \quad \partial_\nu \Delta u = 0 \quad \text{in } \partial\Omega.$$

4 The second domain variation

Throughout this chapter we assume that Ω is an optimal domain, i.e. $\dot{\Lambda}(0) = 0$ and $\ddot{\Lambda}(0) \geq 0$. This implies that u solves (3.5) - (3.7) and (3.8). As a consequence (2.14) reads as

$$(4.1) \quad u'(x) = 0 \quad \text{and} \quad \partial_\nu u'(x) = -c_0 v(x) \cdot \nu(x) \quad \text{for } x \in \partial\Omega.$$

Note that if we differentiate (2.6) - (2.7) in $t = 0$ and use the fact that $\dot{\Lambda}(0) = 0$, we obtain an equation for u' :

$$(4.2) \quad \Delta^2 u'(x) + \Lambda(\Omega) \Delta u'(x) = 0 \quad \text{in } \Omega.$$

The boundary conditions for u' are given by (4.1). Furthermore, the normalization (3.1) implies

$$(4.3) \quad \int_{\Omega} \nabla u \cdot \nabla u' \, dx = 0.$$

We recall formula (3.2). Before we differentiate with respect to t again we state the following consequence of Reynold's theorem (see e.g. Chapter 5.4.2 in [4]).

Theorem 3 *Let Ω be a bounded smooth domain of class C^3 . Let $t \in (-t_0, t_0)$ and let $\Phi_t \in C^{0,1}(\mathbb{R}^n)$ be differentiable in t . Let $t \rightarrow g(t) \in L^1(\mathbb{R}^n)$ be a function which is differentiable in t . Moreover, let $g(t) \in W^{1,1}(\mathbb{R}^n)$. Then $t \rightarrow J(t) := \int_{\partial\Omega_t} g(t) \, dS(y)$ is differentiable in t . Moreover, for $t = 0$ we have the formula*

$$\dot{J}(0) = \int_{\partial\Omega} \partial_t g(0) + (v(x) \cdot \nu(x)) \{ \partial_\nu g(0) + (n-1)g(0) H_{\partial\Omega}(x) \} \, dS(x),$$

where $H_{\partial\Omega}$ denotes the mean curvature of $\partial\Omega$ in x .

We apply this theorem to (3.2). It is convenient to apply (2.17) and to rewrite (3.2) as

$$\dot{\Lambda}(t) = \int_{\partial\Omega_t} \Delta \tilde{u}(t) \nu_t \cdot \nabla(\partial_t u(t, y)) \, dS(y).$$

Let

$$g(t) := \Delta \tilde{u}(t) \nu_t \cdot \nabla(\partial_t u(t, y)).$$

An application of Theorem 3 yields

$$(4.4) \quad \begin{aligned} \ddot{\Lambda}(0) &= \int_{\partial\Omega} \Delta u' \partial_\nu u' \, dS + \int_{\partial\Omega} \Delta u \nu' \cdot \nabla u' \, dS + \int_{\partial\Omega} \Delta u \partial_\nu u'' \, dS \\ &\quad + \int_{\partial\Omega} (v \cdot \nu) \partial_\nu (\Delta u \partial_\nu u') \, dS + (n-1) \int_{\partial\Omega} (v \cdot \nu) \Delta u \partial_\nu u' H_{\partial\Omega} \, dS. \end{aligned}$$

Note that

$$\nu_t \cdot \nu_t = 1 \text{ in } \partial\Omega_t \implies \nu \cdot \nu' = 0 \text{ in } \partial\Omega,$$

where

$$\nu'(x) = \partial_t \nu(t, \Phi_t(x))|_{t=0} \text{ for } x \in \partial\Omega.$$

Since (4.1) implies $\nabla u' = \partial_\nu u' \nu$, this implies

$$\int_{\partial\Omega} \Delta u \nu' \cdot \nabla u' dS = 0.$$

For the fourth integral we apply (3.4) and (3.9).

$$\begin{aligned} \int_{\partial\Omega} (v \cdot \nu) \partial_\nu (\Delta u \partial_\nu u') dS &= \int_{\partial\Omega} (v \cdot \nu) \partial_\nu \Delta u \partial_\nu u' dS + \int_{\partial\Omega} (v \cdot \nu) \Delta u \partial_\nu^2 u' dS \\ &= 0 + c_0 \int_{\partial\Omega} (v \cdot \nu) \partial_\nu^2 u' dS. \end{aligned}$$

With the help of (4.1) and (2.8) we write

$$\partial_\nu^2 u' = \Delta u' - (n-1) \partial_\nu u' H_{\partial\Omega}.$$

Hence,

$$\int_{\partial\Omega} (v \cdot \nu) \partial_\nu (\Delta u \partial_\nu u') dS = c_0 \int_{\partial\Omega} (v \cdot \nu) \Delta u' dS - c_0(n-1) \int_{\partial\Omega} (v \cdot \nu) \partial_\nu u' H_{\partial\Omega} dS.$$

Our computations yield a first simplification of (4.4):

$$\ddot{\Lambda}(0) = \int_{\partial\Omega} \Delta u' \partial_\nu u' dS + \int_{\partial\Omega} \Delta u \partial_\nu u'' dS + c_0 \int_{\partial\Omega} (v \cdot \nu) \Delta u' dS.$$

In the first integral on the right hand side we use (4.1) again. Thus, we get

$$(4.5) \quad \ddot{\Lambda}(0) = c_0 \int_{\partial\Omega} \partial_\nu u'' dS$$

In order to find a lower bound for $\ddot{\Lambda}(0)$, we analyze the integral in (4.5). Recall (2.18). We differentiate this equation with respect to t in $t = 0$. Then (3.9) and (3.4) yield

$$\begin{aligned} \nu' \cdot \nabla u' + v \cdot D\nu \cdot \nabla u' + \partial_\nu u'' + \nu \cdot D^2 u' \cdot v = \\ -\Delta u' (v \cdot \nu) - c_0 (v \cdot \nu') - c_0 v \cdot D\nu \cdot v - c_0 (w \cdot \nu). \end{aligned}$$

As before, $\nu' \cdot \nabla u' = 0$ on $\partial\Omega$. Moreover, by (4.1)

$$v \cdot D\nu \cdot \nabla u' = -c_0 v \cdot D\nu \cdot \nu (v \cdot \nu) = 0,$$

where the last equality follows from (2.1). Thus,

$$(4.6) \quad \begin{aligned} \ddot{\Lambda}(0) &= -c_0 \int_{\partial\Omega} (v \cdot \nu) \Delta u' dS - c_0 \int_{\partial\Omega} \nu \cdot D^2 u' \cdot v dS \\ &\quad - c_0^2 \int_{\partial\Omega} (v \cdot \nu') dS - c_0^2 \int_{\partial\Omega} v \cdot D\nu \cdot v dS - c_0^2 \int_{\partial\Omega} (w \cdot \nu) dS. \end{aligned}$$

For the first integral we use (4.1) and we observe that Gauß theorem, partial integration and equation (4.2) for u' gives

$$(4.7) \quad -c_0 \int_{\partial\Omega} (v \cdot \nu) \Delta u' dS = \int_{\partial\Omega} \Delta u' \partial_\nu u' dS = \int_{\Omega} |\Delta u'|^2 dx - \Lambda(\Omega) \int_{\Omega} |\nabla u'|^2 dx.$$

The second integral is slightly more involved. We set $v^\tau = v - (v \cdot \nu)\nu$. Since $\nabla u' = (\partial_\nu u')\nu$ and since (2.8) can be applied to u' , we get

$$\begin{aligned} -c_0 \int_{\partial\Omega} v \cdot D^2 u' \cdot \nu dS &= -c_0 \int_{\partial\Omega} v^\tau \cdot D^2 u' \cdot \nu dS - c_0 \int_{\partial\Omega} (v \cdot \nu) (\Delta u' - (n-1)\partial_\nu u' H_{\partial\Omega}) dS \\ &= -c_0 \int_{\partial\Omega} v^\tau \cdot D(\partial_\nu u' \nu) \cdot \nu dS - c_0 \int_{\partial\Omega} (v \cdot \nu) \Delta u' dS \\ &\quad - c_0^2 (n-1) \int_{\partial\Omega} (v \cdot \nu)^2 H_{\partial\Omega} dS. \end{aligned}$$

For the last equality we also used

$$v^\tau \cdot D\nu \cdot \nu = v^\tau \cdot D^\tau \nu \cdot \nu = 0 \quad \text{in } \partial\Omega.$$

Next we note that with (4.1) we have

$$\begin{aligned} -c_0 \int_{\partial\Omega} v^\tau \cdot D(\partial_\nu u' \nu) \cdot \nu dS &= -c_0 \int_{\partial\Omega} v^\tau \cdot D^\tau (\partial_\nu u' \nu) \cdot \nu dS = c_0^2 \int_{\partial\Omega} v^\tau \cdot D^\tau ((v \cdot \nu) \nu) \cdot \nu dS \\ &= c_0^2 \int_{\partial\Omega} v^\tau \cdot \nabla^\tau (v \cdot \nu) dS, \end{aligned}$$

where the last equality uses (2.1).

For the third integral in (4.6) we apply formula (2.16):

$$-c_0^2 \int_{\partial\Omega} (v \cdot \nu') dS = c_0^2 \int_{\partial\Omega} v \cdot \nabla^\tau (v \cdot \nu) dS = c_0^2 \int_{\partial\Omega} v^\tau \cdot \nabla^\tau (v \cdot \nu) dS.$$

These computations simplify (4.6) and we obtain

$$(4.8) \quad \ddot{\Lambda}(0) = 2 \int_{\partial\Omega} \partial_\nu u' \Delta u' dS + 2c_0^2 \int_{\partial\Omega} v^\tau \cdot \nabla^\tau (v \cdot \nu) dS - c_0^2 (n-1) \int_{\partial\Omega} (v \cdot \nu)^2 H_{\partial\Omega} dS \\ - c_0^2 \int_{\partial\Omega} v \cdot D\nu \cdot v dS - c_0^2 \int_{\partial\Omega} (w \cdot \nu) dS.$$

Next we use the volume constraint (2.21).

$$\begin{aligned} -c_0^2 \int_{\partial\Omega} (w \cdot \nu) dS &= c_0^2 \int_{\partial\Omega} (v \cdot \nu) \operatorname{div} v dS - c_0^2 \int_{\partial\Omega} v \cdot Dv \cdot \nu dS \\ &= c_0^2 \int_{\partial\Omega} (v \cdot \nu) \operatorname{div}_{\partial\Omega} v dS - c_0^2 \int_{\partial\Omega} v^\tau \cdot D^\tau v \cdot \nu dS. \end{aligned}$$

We integrate by parts in the first integral (see formula (2.3) and (2.4)).

$$\begin{aligned} -c_0^2 \int_{\partial\Omega} (w \cdot \nu) dS &= -c_0^2 \int_{\partial\Omega} v^\tau \cdot D^\tau (v \cdot \nu) dS + c_0^2 (n-1) \int_{\partial\Omega} (v \cdot \nu)^2 H_{\partial\Omega} dS \\ &\quad - c_0^2 \int_{\partial\Omega} v^\tau \cdot D^\tau v \cdot \nu dS. \end{aligned}$$

Thus, (4.8) becomes

$$\begin{aligned} \ddot{\Lambda}(0) &= 2 \int_{\partial\Omega} \partial_\nu u' \Delta u' dS + c_0^2 \int_{\partial\Omega} v^\tau \cdot \nabla^\tau (v \cdot \nu) dS - c_0^2 \int_{\partial\Omega} v^\tau \cdot D^\tau v \cdot \nu dS \\ &\quad - c_0^2 \int_{\partial\Omega} v \cdot D\nu \cdot v dS. \end{aligned}$$

An application of (2.1) and (2.16) yields

$$\begin{aligned} v^\tau \cdot \nabla^\tau (v \cdot \nu) - v^\tau \cdot D^\tau v \cdot \nu - v \cdot D\nu \cdot v &= v^\tau \cdot D^\tau \nu \cdot v - v \cdot D\nu \cdot v \\ &= -(v \cdot \nu) \nu \cdot D\nu \cdot v = 0. \end{aligned}$$

Thus, with (4.8) we proved the following lemma.

Lemma 4 *Let u' be the shape derivative of u resulting from a volume preserving perturbation of Ω . Then there holds*

$$\frac{\ddot{\Lambda}(0)}{2} = \mathcal{E}(u'),$$

where

$$\mathcal{E}(u') = \int_{\Omega} |\Delta u'|^2 dx - \Lambda(\Omega) \int_{\Omega} |\nabla u'|^2 dx.$$

5 Minimization of the second domain variation

In this chapter we consider the quadratic functional

$$(5.1) \quad \mathcal{E}(\varphi) := \int_{\Omega} |\Delta \varphi|^2 dx - \Lambda(\Omega) \int_{\Omega} |\nabla \varphi|^2 dx$$

for $\varphi \in H_0^{1,2} \cap H^{2,2}(\Omega)$. It will be convenient to work with an alternative representation of \mathcal{E} . For $\varphi \in H_0^{1,2} \cap H^{2,2}(\Omega)$ there holds

$$\mathcal{E}(\varphi) = \int_{\Omega} |D^2 \varphi|^2 - \Lambda(\Omega) |\nabla \varphi|^2 dx + \int_{\partial \Omega} \Delta \varphi \partial_{\nu} \varphi - \varphi \cdot D^2 \varphi \cdot \nu dS.$$

We apply (2.8) and (2.1).

$$\begin{aligned} \Delta \varphi \partial_{\nu} \varphi - \varphi \cdot D^2 \varphi \cdot \nu &= \partial_{\nu}^2 \varphi \partial_{\nu} \varphi + (n-1)(\partial_{\nu} \varphi)^2 H_{\partial \Omega} - \varphi \cdot D^2 \varphi \cdot \nu \\ &= \nu \cdot D^2 \varphi \cdot \nu (\nu \cdot \nabla \varphi) + (n-1)(\partial_{\nu} \varphi)^2 H_{\partial \Omega} - \varphi \cdot D^2 \varphi \cdot \nu \\ &= (n-1)(\partial_{\nu} \varphi)^2 H_{\partial \Omega}. \end{aligned}$$

Consequently, we get

$$(5.2) \quad \mathcal{E}(\varphi) = \int_{\Omega} |D^2 \varphi|^2 dx - \Lambda(\Omega) \int_{\Omega} |\nabla \varphi|^2 dx + (n-1) \int_{\partial \Omega} (\partial_{\nu} \varphi)^2 H_{\partial \Omega} dS.$$

Remark 5 *The functional \mathcal{E} is lower semicontinuous with respect to weak convergence in $H_0^{1,2} \cap H^{2,2}(\Omega)$.*

Since Ω is optimal, we know from Lemma 4 that

$$\mathcal{E}(\varphi) \geq 0$$

for all φ which are shape derivatives of u . Recall that φ is a shape derivative, if it solves (1.6) - (1.9) for some vector field v in the class described in Chapter 1 (Lemma 2).

The following remark shows a property of shape derivatives we have not yet mentioned.

Remark 6 *Let φ be a shape derivative and assume that $\partial_{\nu} \varphi \equiv 0$ in $\partial \Omega$. Then $\varphi \in H_0^{2,2}(\Omega)$ and, since φ satisfies equation (4.2), φ is a buckling eigenfunction in Ω . Thus by uniqueness of u we get $\varphi = \alpha u$ for any $\alpha \in \mathbb{R}$. Then formula (1.4) yields*

$$\Lambda(\Omega) = \int_{\partial \Omega} |\Delta \varphi|^2 x \cdot \nu dS = \alpha^2 c_o^2 \int_{\partial \Omega} x \cdot \nu dS = \alpha^2 \int_{\partial \Omega} |\Delta u|^2 x \cdot \nu dS = \alpha^2 \Lambda(\Omega).$$

Thus, $\alpha^2 = 1$ and there holds

$$\left| \int_{\Omega} \nabla u \cdot \nabla \varphi \, dx \right| = 1.$$

This is contradictory to (4.3) and thus $\partial_{\nu} \varphi$ cannot vanish identically on $\partial\Omega$.

This motivates the following definition.

$$\mathcal{Z} := \left\{ \varphi \in H_0^{1,2} \cap H^{2,2}(\Omega) : \int_{\partial\Omega} \partial_{\nu} \varphi \, dS = 0, \int_{\partial\Omega} (\partial_{\nu} \varphi)^2 \, dS > 0, \int_{\Omega} \nabla u \cdot \nabla \varphi \, dx = 0 \right\}.$$

Note that \mathcal{Z} contains elements which are not shape derivatives. Nevertheless we will show that

$$\mathcal{E}|_{\mathcal{Z}} \geq 0.$$

The next lemma ensures that \mathcal{Z} is not empty and that at least for a specific shape derivative \mathcal{E} is equal to zero.

Lemma 5 *For each $1 \leq k \leq n$ the directional derivative $\partial_k u$ satisfies $\partial_k u \in \mathcal{Z}$. Furthermore, $\mathcal{E}(\partial_k u) = 0$.*

Proof Let $1 \leq k \leq n$. Due to (1.2) and (1.3) $\partial_k u$ satisfies

$$(5.3) \quad \begin{aligned} \Delta^2 \partial_k u + \Lambda(\Omega) \Delta \partial_k u &= 0 & \text{in } \Omega \\ \partial_k u &= 0 & \text{in } \partial\Omega. \end{aligned}$$

According to (2.9) there holds $\partial_{\nu} \partial_k u = c_0 \nu_k$ on $\partial\Omega$. Hence,

$$\int_{\partial\Omega} \partial_{\nu} \partial_k u \, dS = c_0 \int_{\partial\Omega} \nu_k \, dS = 0.$$

In addition, we find that

$$\int_{\Omega} \nabla u \cdot \nabla \partial_k u \, dx = \frac{1}{2} \int_{\partial\Omega} |\nabla u|^2 \nu_k \, dS = 0.$$

Following the idea of Remark 6, we obtain that $\partial_{\nu} \partial_k u$ does not vanish identically on $\partial\Omega$. Thus, $\partial_k u \in \mathcal{Z}$. Moreover, (3.9) and (5.3) imply

$$\mathcal{E}(\partial_k u) = \int_{\Omega} (\Delta^2 \partial_k u + \Lambda(\Omega) \Delta \partial_k u) \partial_k u \, dx + \int_{\partial\Omega} \partial_k \Delta u \, \partial_{\nu} \partial_k u \, dS = 0.$$

This proves the lemma. □

Note that each directional derivative of u is a shape derivative resulting from translations of Ω .

Theorem 4 *The infimum of the functional \mathcal{E} in \mathcal{Z} is finite.*

Proof We argue by contradiction. Let us assume that $\inf_{\mathcal{Z}} \mathcal{E} = -\infty$ and consider a sequence $(\hat{w}_k)_k \subset \mathcal{Z}$ such that

$$\lim_{k \rightarrow \infty} \mathcal{E}(\hat{w}_k) = -\infty.$$

For this sequence there either holds

$$\int_{\partial\Omega} (\partial_\nu \hat{w}_k)^2 dS \xrightarrow{k \rightarrow \infty} 0 \quad \text{or} \quad \int_{\partial\Omega} (\partial_\nu \hat{w}_k)^2 dS \not\xrightarrow{k \rightarrow \infty} 0.$$

If the second case holds true, we normalize the sequence $(\hat{w}_k)_k$ such that $\|\partial_\nu \hat{w}_k\|_{L^2(\partial\Omega)} = 1$. Hence, in either case, for each \hat{w}_k there holds $\|\partial_\nu \hat{w}_k\|_{L^2(\partial\Omega)} \leq 1$. Thus, (2.2) gives

$$\left| \int_{\partial\Omega} H_{\partial\Omega} (\partial_\nu \hat{w}_k)^2 dS \right| \leq \max_{\partial\Omega} |H_{\partial\Omega}| < \infty.$$

We use (5.2) and obtain

$$(5.4) \quad \mathcal{E}(\hat{w}_k) \geq -\Lambda(0) \int_{\Omega} |\nabla \hat{w}_k| dx - (n-1) \max_{\partial\Omega} |H_{\partial\Omega}|.$$

The assumption $\lim_{k \rightarrow \infty} \mathcal{E}(\hat{w}_k) = -\infty$ implies

$$\int_{\Omega} |\nabla \hat{w}_k|^2 dx \xrightarrow{k \rightarrow \infty} \infty.$$

We define

$$w_k := \frac{1}{\|\nabla \hat{w}_k\|_{L^2(\Omega)}} \hat{w}_k.$$

Then there holds

$$(5.5) \quad \|\nabla w_k\|_{L^2(\Omega)} = 1 \quad \text{and} \quad \int_{\partial\Omega} (\partial_\nu w_k)^2 dS \xrightarrow{k \rightarrow \infty} 0.$$

Moreover, for each $k \in \mathbb{N}$ estimate (5.4) implies

$$\mathcal{E}(w_k) \geq -\Lambda(0) - C$$

and the infimum of \mathcal{E} in $M := \{w_k : k \in \mathbb{N}\}$ is finite. Therefore, we can choose a subsequence of $(w_k)_k$, denote by $(w_k)_k$ as well, such that

$$\lim_{k \rightarrow \infty} \mathcal{E}(w_k) = \inf_M \mathcal{E}.$$

Now Poincaré's inequality and the previous estimates imply

$$\begin{aligned}
\|w_k\|_{H^{2,2}(\Omega)}^2 &= \int_{\Omega} |D^2 w_k|^2 + |\nabla w_k|^2 + w_k^2 dx \\
&\leq \mathcal{E}(w_k) + C \int_{\Omega} |\nabla w_k|^2 dx + (n-1) \int_{\partial\Omega} |H_{\partial\Omega}| (\partial_{\nu} w_k)^2 dS \\
&\leq C.
\end{aligned}$$

Thus, the sequence $(w_k)_k$ is uniformly bounded in $H^{2,2}(\Omega)$ and there exists a $w \in H^{2,2}(\Omega)$ such that $(w_k)_k$ weakly converges to w . In view of (5.5), the limit function w satisfies $\|\nabla w\|_{L^2(\Omega)} = 1$ and $\partial_{\nu} w = 0$ on $\partial\Omega$. Since $w_k = 0$ in $\partial\Omega$ for each $k \in \mathbb{N}$, we conclude that $w \in H_0^{2,2}(\Omega)$. Now let us recall that $\mathcal{E}(\hat{w}_k)$ converges to $-\infty$. Thus there exists a $k_0 \in \mathbb{N}$ such that

$$\mathcal{E}(w_k) = \frac{1}{\|\nabla \hat{w}_k\|_{L^2(\Omega)}} \mathcal{E}(\hat{w}_k) < 0$$

for all $k \geq k_0$. Since the functional \mathcal{E} is lower semicontinuous with respect to weak convergence in $H^{2,2}(\Omega)$, we find that $\mathcal{E}(w) < 0$. According to the definition of \mathcal{E} in (5.1), this immediately leads to

$$\frac{\int_{\Omega} |\Delta w|^2 dx}{\int_{\Omega} |\nabla w|^2 dx} < \Lambda(\Omega).$$

Since $w \in H_0^{2,2}(\Omega)$ this is contradictory to the minimum property of $\Lambda(\Omega)$. \square

As mentioned in the previous proof, a minimizing sequence $(\varphi_k)_k \subset \mathcal{Z}$ satisfies one of the following two conditions

- i) $\int_{\partial\Omega} (\partial_{\nu} \varphi_k)^2 dS \xrightarrow{k \rightarrow \infty} 0$
- ii) $\int_{\partial\Omega} (\partial_{\nu} \varphi_k)^2 dS \not\xrightarrow{k \rightarrow \infty} 0$.

In the sequel, we show that the case i) implies that the minimizing sequence $(\varphi_k)_k$ converges to zero. For this purpose, let $(\varphi_k)_k \subset \mathcal{Z}$ be a minimizing sequence which satisfies condition i). From (5.2) we get

$$\|\varphi_k\|_{H^{2,2}(\Omega)}^2 \leq C,$$

and thus there exists a $\varphi \in H^{2,2}(\Omega)$ such that φ_k weakly converges to φ in $H^{2,2}(\Omega)$ and $\mathcal{E}(\varphi) = \inf_{\mathcal{Z}} \mathcal{E}$. Furthermore, condition i) implies $\varphi \in H_0^{2,2}(\Omega)$. From Lemma 5 we obtain

$$\inf_{\mathcal{Z}} \mathcal{E} = \mathcal{E}(\varphi) \leq \mathcal{E}(\partial_l u) = 0 \quad \text{for any } 1 \leq l \leq n.$$

Hence,

$$\frac{\int_{\Omega} |\Delta \varphi|^2 dx}{\int_{\Omega} |\nabla \varphi|^2 dx} \leq \Lambda(\Omega).$$

Thus φ is necessarily an eigenfunction corresponding to $\Lambda(\Omega)$. Since the eigenvalue is simple we have $\varphi = \alpha u$ for $\alpha \in \mathbb{R}$. Now let us recall that $\varphi_k \in \mathcal{Z}$ and, therefore,

$$\alpha = \alpha \int_{\Omega} |\nabla u|^2 dx = \int_{\Omega} \nabla u \cdot \nabla \varphi dx = \lim_{k \rightarrow \infty} \int_{\Omega} \nabla u \cdot \nabla \varphi_k dx = 0.$$

Consequently, $\alpha = 0$ and $\varphi \equiv 0$ in Ω . Hence $\varphi \notin \mathcal{Z}$. Since we are interested to find minimizers of \mathcal{E} in \mathcal{Z} , we restrict ourselves to minimizing sequences which satisfy the condition ii). Thus we consider the functional

$$\tilde{\mathcal{E}}(\varphi) := \frac{\mathcal{E}(\varphi)}{\int_{\partial\Omega} (\partial_{\nu} \varphi)^2 dS},$$

where $\varphi \in \mathcal{Z}$ and we set $\tilde{\mathcal{E}} = \infty$ if $\int_{\partial\Omega} (\partial_{\nu} \varphi_k)^2 dS = 0$.

Remark 7 Suppose $(\varphi_k)_k \subset \mathcal{Z}$ is a minimizing sequence for $\tilde{\mathcal{E}}$ in \mathcal{Z} . Then there exists a constant $C > 0$ such that $\|\nabla \varphi_k\|_{L^2(\Omega)} \leq C$ for every $k \in \mathbb{N}$. This follows by contradiction. Otherwise we may assume that $\|\nabla \varphi_k\|_{L^2(\Omega)}$ tends to infinity as $k \rightarrow \infty$, we define $\varphi_k^* := \|\nabla \varphi_k\|_{L^2(\Omega)}^{-1} \varphi_k$. Then $(\varphi_k^*)_k$ is uniformly bounded in $H^{2,2}(\Omega)$ and

$$\int_{\partial\Omega} (\partial_{\nu} \varphi_k^*)^2 dS \xrightarrow{k \rightarrow \infty} 0.$$

Thus, $(\varphi_k^*)_k$ converges weakly to a function $\varphi \in H_0^{2,2}(\Omega)$ and for every $1 \leq l \leq n$ there holds

$$\inf_{\mathcal{Z}} \tilde{\mathcal{E}} = \tilde{\mathcal{E}}(\varphi) \leq \tilde{\mathcal{E}}(\partial_l u) = 0.$$

As the previous considerations have shown, this implies $\varphi \equiv 0$ in Ω . Thus, our assumption cannot be true.

We now consider a minimizing sequence $(\varphi_k)_k \subset \mathcal{Z}$ which satisfies

$$(5.6) \quad \int_{\partial\Omega} (\partial_{\nu} \varphi_k)^2 dS = 1$$

for all $k \in \mathbb{N}$. As before we obtain the inequality

$$\|\varphi_k\|_{H^{2,2}(\Omega)}^2 \leq \mathcal{E}(\varphi_k) + C \int_{\Omega} |\nabla \varphi_k|^2 dx.$$

Thus, $(\varphi_k)_k$ is uniformly bounded in $H^{2,2}(\Omega)$ and φ_k converges weakly to a $\varphi^* \in H^{2,2}(\Omega)$. We find that $\varphi^* \in \mathcal{Z}$ and $\mathcal{E}(\varphi^*) = \inf_{\mathcal{Z}} E$. In addition, there holds

$$\int_{\partial\Omega} (\partial_\nu \varphi^*)^2 dS = 1.$$

Hence, φ^* minimizes $\tilde{\mathcal{E}}$ in \mathcal{Z} . Suppose $\theta \in \mathcal{Z}$, then the minimality of φ^* implies

$$\left. \frac{d}{dt} \frac{\mathcal{E}(\varphi^* + t\theta)}{\int_{\partial\Omega} (\partial_\nu(\varphi^* + t\theta))^2 dS} \right|_{t=0} = 0$$

and we obtain

$$\int_{\Omega} [\Delta^2 \varphi^* + \Lambda(\Omega) \Delta \varphi^*] \theta dx - \int_{\partial\Omega} [\Delta \varphi^* + \rho \partial_\nu \varphi^*] \partial_\nu \theta dS = 0.$$

Since $\theta \in \mathcal{Z}$ was chosen arbitrary, φ^* satisfies the Euler-Lagrange equalities

$$\begin{aligned} \Delta^2 \varphi^* + \Lambda(\Omega) \Delta \varphi^* &= 0 \quad \text{in } \Omega \\ \Delta \varphi^* + \rho \partial_\nu \varphi^* &= \text{const.} \quad \text{in } \partial\Omega, \end{aligned}$$

where $\rho := \min_{\mathcal{Z}} \tilde{\mathcal{E}}$. The following theorem collects the previous results.

Theorem 5 *There exists a function $\varphi^* \in \mathcal{Z}$ such that $\tilde{\mathcal{E}}(\varphi^*) = \min_{\mathcal{Z}} \tilde{\mathcal{E}}$. Furthermore, any minimizer $\varphi^* \in \mathcal{Z}$ satisfies*

$$\begin{aligned} (5.7) \quad \Delta^2 \varphi^* + \Lambda(\Omega) \Delta \varphi^* &= 0 \quad \text{in } \Omega \\ (5.8) \quad \Delta \varphi^* + \rho \partial_\nu \varphi^* &= \text{const.} \quad \text{in } \partial\Omega \\ \varphi^* &= 0 \quad \text{in } \partial\Omega, \end{aligned}$$

where $\rho := \min_{\mathcal{Z}} \tilde{\mathcal{E}}$.

The next theorem shows that in fact $\rho = 0$.

Theorem 6 *Suppose $\varphi^* \in \mathcal{Z}$ is a minimizer of $\tilde{\mathcal{E}}$. Then there holds $\tilde{\mathcal{E}}(\varphi^*) = 0$. In particular, $\mathcal{E} \geq 0$ in \mathcal{Z} .*

Proof Let $\varphi^* \in \mathcal{Z}$ be a minimizer of $\tilde{\mathcal{E}}$. Since φ^* satisfies equation (5.7) and $\partial\Omega$ is smooth, φ^* is a smooth function on $\overline{\Omega}$. Hence, we may define a volume preserving perturbation Φ_t of Ω such that

$$\partial_\nu u'(x) = \partial_\nu \varphi^*(x) \quad \text{for } x \in \partial\Omega.$$

Note that this can be achieved by setting $v = c_0^{-1} \nabla \varphi^*$ in $\partial\Omega$. In this way, each minimizer φ^* implies the existence of vector fields v and w in the sense of Section 2. We define $\psi := u' - \varphi^*$, then $\psi \in H_0^{2,2}(\Omega)$ and

$$\Delta^2 \psi + \Lambda(\Omega) \Delta \psi = 0 \quad \text{in } \Omega.$$

The uniqueness of u implies $\psi = \alpha u$ for an $\alpha \in \mathbb{R}$. Since $\varphi^* \in \mathcal{Z}$, equation (4.3) yields

$$0 = \int_{\Omega} \nabla u \cdot \nabla u' \, dx - \int_{\Omega} \nabla u \cdot \nabla \varphi^* \, dx = \int_{\Omega} \nabla u \cdot \nabla \psi \, dx = \alpha.$$

Consequently, $u' \equiv \varphi^*$. Thus φ^* is a shape derivative. Since Ω is optimal $\tilde{\mathcal{E}}(\varphi^*) \geq 0$. Finally we apply Lemma 5. This gives

$$0 \leq \tilde{\mathcal{E}}(\varphi^*) = \min_{\mathcal{Z}} \tilde{\mathcal{E}} \leq \tilde{\mathcal{E}}(\partial_k u) = 0.$$

□

6 The optimal domain is a ball

We will use an inequality due to L.E. Payne to show that the optimal domain Ω is a ball. Payne's inequality (see [13]) states that for each domain G there holds

$$\lambda_2(G) \leq \Lambda(G)$$

and equality only holds if and only if G is a ball. Thereby λ_2 denotes the second Dirichlet eigenfunction of the Laplacian. In the sequel, we construct a suitable function $\psi \in \mathcal{Z}$ such that the condition $\mathcal{E}(\psi) \geq 0$ (due to Theorem 6) will imply that the optimal domain Ω is a ball. For this purpose, we denote by u_1 and u_2 the first and the second Dirichlet eigenfunction for the Laplacian in Ω . Thus, for $1 \leq k \leq 2$ there holds

$$\begin{aligned} \Delta u_k + \lambda_k(\Omega) u_k &= 0 \quad \text{in } \Omega \\ u_k &= 0 \quad \text{in } \partial\Omega, \end{aligned}$$

where $\lambda_k(\Omega)$ is the k -th Dirichlet eigenvalue for the Laplacian in Ω . Note that $0 < \lambda_1(\Omega) < \lambda_2(\Omega)$. For the sake of brevity, we will write λ_k instead of $\lambda_k(\Omega)$ and Λ instead of $\Lambda(\Omega)$. In addition, we assume $\|u_k\|_{L^2(\Omega)} = 1$ and

$$\int_{\Omega} u_1 u_2 \, dx = 0.$$

Without loss of generality, we may assume that

$$\int_{\Omega} u_1 \, dx > 0 \quad \text{and} \quad \int_{\Omega} u_2 \, dx \leq 0.$$

Consequently, there exists a $t \in (0, 1]$ such that

$$(6.1) \quad \int_{\Omega} (1-t) \lambda_1 u_1 + t \lambda_2 u_2 \, dx = 0.$$

This fixes t . Next we define

$$\psi(x) := (1-t)u_1(x) + t u_2(x) + c u(x) \quad \text{for } x \in \overline{\Omega},$$

where u is the first buckling eigenfunction in Ω . The constant c is given by

$$c := -\frac{1}{\Lambda} \int_{\Omega} (1-t)\lambda_1 \nabla u \cdot \nabla u_1 + t\lambda_2 \nabla u \cdot \nabla u_2 \, dx.$$

In a first step we show that $\psi \in \mathcal{Z}$. Note that $\psi \in H_0^{1,2} \cap H^{2,2}(\Omega)$. Moreover the definition of ψ , the fact that $\partial_\nu u = 0$ on $\partial\Omega$, the equations for u_1 and u_2 , and (6.1) imply

$$\int_{\partial\Omega} \partial_\nu \psi \, dS = \int_{\Omega} (1-t) \Delta u_1 + t \Delta u_2 \, dx = - \int_{\Omega} (1-t)\lambda_1 u_1 + t\lambda_2 u_2 \, dx = 0.$$

By the unique continuation principle $\partial_\nu \psi$ does not vanish identically in $\partial\Omega$. Thus, to show that $\psi \in \mathcal{Z}$, it remains to prove that

$$(6.2) \quad \int_{\Omega} \nabla u \cdot \nabla \psi \, dx = 0.$$

We recall that $\Delta u = c_0$ in $\partial\Omega$. Hence

$$\begin{aligned} 0 &= \int_{\Omega} (\Delta^2 u + \Lambda \Delta u) \psi \, dx = \int_{\Omega} \Delta u \Delta \psi \, dx - \Lambda \int_{\Omega} \nabla u \cdot \nabla \psi \, dx \\ &= - \int_{\Omega} [(1-t)\lambda_1 u_1 + t\lambda_2 u_2] \Delta u \, dx + c \int_{\Omega} |\Delta u|^2 \, dx - \Lambda \int_{\Omega} \nabla u \cdot \nabla \psi \, dx. \end{aligned}$$

Since $\|\nabla u\|_{L^2(\Omega)} = 1$, the second integral is equal to Λ . Thus, the definition of c implies (6.2). Note that ψ is not a shape derivative since it fails to satisfy (4.2) - unless $t = 1$ and Ω equals a ball. However, $\psi \in \mathcal{Z}$ and, according to Theorem 6, there holds $\tilde{\mathcal{E}}(\psi) \geq 0$. Consequently, $\mathcal{E}(\psi) \geq 0$. Thus

$$\begin{aligned} \mathcal{E}(\psi) &= \int_{\Omega} |\Delta \psi|^2 - \Lambda |\nabla \psi|^2 \, dx \\ &= (1-t)^2 \lambda_1 (\lambda_1 - \Lambda) + t^2 \lambda_2 (\lambda_2 - \Lambda) + 2c c_0 \int_{\Omega} (1-t)\lambda_1 u_1 + t\lambda_2 u_2 \, dx \\ &\stackrel{(6.1)}{=} (1-t)^2 \lambda_1 (\lambda_1 - \Lambda) + t^2 \lambda_2 (\lambda_2 - \Lambda) \geq 0. \end{aligned}$$

Since $\lambda_1 - \Lambda < 0$ and $\lambda_2 - \Lambda \leq 0$, both summands in $\mathcal{E}(\psi)$ have to vanish. Consequently $t = 1$ and $\lambda_2(\Omega) = \Lambda(\Omega)$. Payne's inequality implies that Ω is a ball. This proves the main theorem of the paper.

Theorem 7 *Let Ω be a bounded, smooth and simply connected domain in \mathbb{R}^n , which minimizes the first buckling eigenvalue among all bounded, smooth and simply connected domains in \mathbb{R}^n with given measure. Then Ω is a ball.*

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